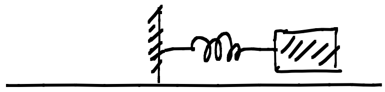
reference: Bates + Weinstein. Lectures on the Geometry of Quantization. 2018 Spring.

§ Harmonic Oscillator

Classical

$m\ddot{x} = -kx$



2^{nd} order ODE $\rightsquigarrow$ 1^{st} order system on "phase space" $(q, p) = (x, m\dot{x})$

$$\Leftrightarrow \quad \begin{cases} \dot{q} = m^{-1} p \\ \dot{p} = -k\, q \end{cases}$$

$\Leftrightarrow$ Hamiltonian equation

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$

$$H = \frac{p^2}{2m} + \frac{k}{2} q^2$$

($\rightsquigarrow$ elliptics centered at $(0,0)$, move clockwise).

- Conservation of energy: $H \equiv \text{const}$ along any sol^{n} curve

  $\left[\because \frac{dH}{dt} = \frac{\partial H}{\partial q} \cdot \dot{q} + \frac{\partial H}{\partial p} \dot{p} = (-\dot{p})\dot{q} + (\dot{q})\dot{p} = 0\right.$

- Area preserving flow on $\mathbb{R}^2$: $\operatorname{div}(X_H) = 0$

  $\left[\because \nabla \cdot X_H = \frac{\partial}{\partial q}(\dot{q}) + \frac{\partial}{\partial p}(\dot{p}) = \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0\right]$

  $X_H = (\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$

Quantum

$\psi(x,t) \in \mathbb{C}$ wave function

$$i\hbar \frac{\partial}{\partial t} \psi = \underbrace{\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{k}{2} x^2\right]}_{\hat{H}} \psi \quad \text{Schrödinger eq}^{t}$$

$\int_{\mathbb{R}} |\psi|^2 = 1$ normalized.

$\hat{H}$: vector field on $\mathcal{H} = C^\infty(\mathbb{R}, \mathbb{C})$

eq^{t}. $\rightsquigarrow$ flow on $\mathcal{H}$.

$|\psi(x,t)|^2$: probability density for observing oscillator at position x, time t.

Classical vs Quantum.

$$H = \frac{1}{2m} p^2 + \frac{k}{2} q^2 \xrightarrow[p \mapsto \hat{p} = -i\hbar \frac{\partial}{\partial x}]{q \mapsto \hat{q} = \cdot x} \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{k}{2} x^2$$

($qp \longmapsto \hat{q}\hat{p}$ or $\hat{p}\hat{q}$? non-comm!)

classical sol^n. $\overset{?}{\rightsquigarrow}$ approx. sol^n to Schrödinger eqt.

§2. WKB method.

Classical $H(q,p) = \frac{1}{2m} p^2 + V(q)$ ← potential

Quantum $i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi$ w/ $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$ (S)

(1) Separation of variables. $\underline{\psi(x,t) = \varphi(x)\, e^{-i\omega t}}$ stationary state ($\because$ $\frac{d\psi}{dt}$, ψ same shape).

(S) $\Rightarrow$ $\hbar\omega\, \varphi(x) \cancel{e^{-i\omega t}} = (\hat{H}\varphi)(x) \cancel{e^{-i\omega t}}$

i.e. time-indep. Schrödinger eqt.

$$\hat{H}\varphi = E\varphi \qquad E = \hbar\omega$$

$\Rightarrow$ Energy = eigenvalue of $\hat{H}$ (discrete/quantized)

(2) 'IF' $V(x) \equiv$ const (free particle).

Try $\varphi(x) = e^{ix\xi}$ ← const.

$$\hat{H}\varphi = E\varphi \iff (\hbar\xi)^2 = 2m(E-V)$$

$$\begin{cases} \text{Case } V < E \text{ (big energy)} \Rightarrow \pm\xi \in \mathbb{R} \Rightarrow \text{bdd sol}^n. \\ \text{Case } V > E \Rightarrow \xi \in i\mathbb{R} \Rightarrow \text{unbound, } \nexists \text{physical interpretation} \end{cases}$$

(3) Idea: V varies w/ x $\Rightarrow$ ξ varies w/ x

$\underline{\varphi(x) = e^{iS(x)/\hbar}}$ $S(x)$: phase function

$$\hat{H}\varphi = \left[\frac{S'^2}{2m} + V - \frac{i\hbar}{2m} S''\right]\varphi$$

$$\hat{H}\varphi = E\varphi \xrightarrow{\text{mod } \hbar} H(x, S'(x)) = E$$

$$\left(\frac{S'^2}{2m} + V - \hbar\frac{i}{2m}S'' = E\right) \Rightarrow S'(x) = \pm\left(2m(E - V(x))\right)^{1/2}$$

Geometrically $H : T^*M \to \mathbb{R}$ $(M=\mathbb{R})$

$S : M \to \mathbb{R}$

must satisfy ($\rightsquigarrow$ 1st order approx.). $\underbrace{L := \mathrm{Graph}(dS)}_{\substack{\text{Lagrangian} \\ \text{(projectable)}}} \subseteq \underbrace{H^{-1}(E)}_{\substack{\text{energy} \\ \text{hypersurface}}} \subset \underbrace{T^*M}_{\text{sympl.}}$

$$\Rightarrow \quad (\hat{H} - E)\, e^{iS(x)/\hbar} = O(\hbar)$$

- Same in higher dim:

$$H(q,p) = \frac{1}{2m} p_i^2 + V(q) \quad \& \quad \hat{H} = -\frac{\hbar^2}{2m}\Delta + V$$

- Hamilton-Jacobi theorem. $\underset{\text{Lagr.}}{L} \subset T^*M \xrightarrow{H} \mathbb{R}$

$$H|_L = \text{const.} \iff X_H \subset T_L \subset T_{T^*M}$$

$$\left(X_H = \sum \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}\right).$$

$$\left(\because \quad \iota_{X_H}\big(\overbrace{\textstyle\sum dp_j \wedge dq_j}^{\omega}\big) = dH \quad \& \quad \omega|_L = 0 \right)$$

§ Try $\varphi(x) = e^{iS(x)/\hbar}\, a(x)$ $\rightsquigarrow$ $\hbar^2$-approx.

$$(\hat{H} - E)\varphi = \frac{-1}{2m}\left[i\hbar\left(a\Delta S + 2\sum_j \frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j}\right) + \hbar^2 \Delta a\right] e^{iS/\hbar}$$

$$\hat{H}\varphi = E\varphi \xrightarrow{\text{mod } \hbar^2} a\Delta S + 2\sum_j \frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j} = 0$$

Homog. transport eqt.

Try $\varphi(x) = e^{iS(x)/\hbar}\left(a_0(x) + a_1(x)\hbar\right)$

$$\hat{H}\varphi = E\varphi \xrightarrow{\text{mod } \hbar^3} a_1\Delta S + 2\sum_j \frac{\partial a_1}{\partial x_j}\frac{\partial S}{\partial x_j} = i\Delta a_0$$

inhomog. transport eqt.

Inductively, $\varphi = e^{iS/\hbar}(a_0 + a_1\hbar + \dots + a_n\hbar^n + \dots\dots)$

$$a_k \Delta S + 2\sum_j \frac{\partial a_k}{\partial x_j}\frac{\partial S}{\partial x_j} = i\,\Delta a_{k-1} \qquad \forall\, k = 0, 1, \dots$$

$$\Longrightarrow \quad \hat{H}\varphi = E\varphi \; + \; O(\hbar^\infty)$$

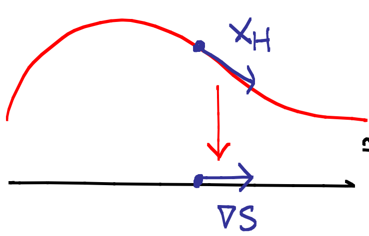
§

$$a\Delta S + 2\sum_j \frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j} = 0$$

$$\Longleftrightarrow \quad \sum_j \frac{\partial}{\partial x_j}\left(a^2 \frac{\partial S}{\partial x_j}\right) = 0$$

$$\Longleftrightarrow \quad a^2\,\nabla S \quad \text{div free v.f. on } M = \mathbb{R}^n$$

i.e. $0 = d\,(\iota_{a^2\nabla S}|dx|) = d\,\iota_{\nabla S}(a^2|dx|) = \mathcal{L}_{\nabla S}(a^2|dx|)$



$$T^*M \xrightarrow{H} \mathbb{R}$$

$L = \mathrm{Graph}(dS)$

$\simeq\downarrow$

M

Note:
$$\begin{array}{ccc} L & & X_H|_L \\ \pi\downarrow\cong & & \pi_*\downarrow ☆ \\ M & & \nabla S \end{array}$$

$$\left[\begin{array}{ll} \because\; X_H|_L = \sum_j \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} & H = \sum \frac{p_j^2}{2} + V(q) \\ \qquad\qquad = \sum_j \frac{\partial S}{\partial x_j}\frac{\partial}{\partial q_j} - \frac{\partial V}{\partial q_j}\frac{\partial}{\partial p_j} & p_j = \frac{\partial S}{\partial q_j} \text{ on } L \\ \pi_* X_H = \sum_j \frac{\partial S}{\partial x_j}\frac{\partial}{\partial x_j} = \nabla S & \end{array}\right]$$

$$\Longleftrightarrow \quad \mathcal{L}_{\pi_* X_H}(a^2|dx|) = 0 \text{ on } \mathbb{R}^n \quad \left(\begin{array}{c}|dx| = |dx_1\wedge\dots\wedge dx_n| \\ \text{canon. density on } \mathbb{R}^n\end{array}\right)$$

$$\Longleftrightarrow \quad \mathcal{L}_{X_H}(\pi^*(a^2|dx|)) = 0 \text{ on } L$$

i.e. $\mathcal{L}_{X_H}(\pi^*(a\,|dx|^{1/2})) = 0$

$\uparrow$ half density

i.e. a s.t. homog. transport eqt.

$\longleftrightarrow$ half density on L, inv. under X_H - flow.

Qu: inhomog. transport eqt $\sim$?

§ Classical Mechanics

$$(M, g) \qquad \begin{array}{c}\text{phase space}\\ T^*M \xrightarrow{K} \mathbb{R} \\ \pi \downarrow \\ M\end{array} \quad \text{w/ } K(q,p) = \tfrac{1}{2}|p|^2 \ \text{Kinetic energy}$$

Theorem: Integral curves of X_K in T^*M $\xrightarrow{\pi}$ geodesics on M (i.e. classical trajectories).

Same for $H(q,p) = \frac{1}{2}|p|^2 + V(q)$

§ Quantum Mechanics

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V \ : C^\infty(M) \circlearrowleft \qquad : \mathcal{H} \circlearrowleft$$

$\mathcal{H}$ = {half density}
$\Downarrow$
Hilbert space str.

Semi-classical approx. $e^{iS/\hbar} a$

w/ (i) $S : M \to \mathbb{R}$ s.t. Hamilton-Jacobi eqt.

i.e. $H \circ \partial_{dS} = E$

(ii) half density a s.t. preserved by $X_H|_L$

i.e. $a \Delta S + 2 L_{\nabla S}\, a = 0$ homog. transport eqt.

$\rightsquigarrow$ 2nd order approx. sol^{n} to Schrödinger eqt. on M.

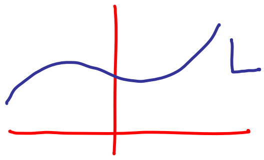
§ Quantization in T^*M

Lagrangian embedding

$$\begin{tikzcd} L \arrow[r, hook, "\iota"] \arrow[d, "\pi_L"] & T^*M \arrow[d, "\pi"] \\ M \arrow[r, phantom, "="] & M \end{tikzcd} \qquad , \quad \omega = d\alpha$$

$$d(\alpha|_L) = \omega|_L = 0 \quad (\because L \text{ Lagr.})$$

$$[\alpha|_L] \in H^1(L, \mathbb{R})$$

Assume: L is a graph  $L \subseteq T^*M$
$\downarrow \pi$
M

(1) If $[\alpha|_L] = 0$

$\Longrightarrow$ $\alpha = d\phi$ $\quad \exists\ M \xleftarrow[\pi]{\sim} L \xrightarrow{\phi} \mathbb{R}$

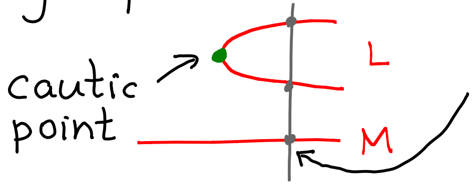
$\rightsquigarrow$ $I_\hbar(L, \iota, a) \triangleq (\pi_L^{-1})^* \, e^{i\phi/\hbar} a$

(2) In fact, $[\alpha|_L] \in H^1(L, 2\pi\hbar\mathbb{Z}) \subset H^1(L, \mathbb{R})$

$\Longrightarrow$ $e^{i\phi/\hbar}$ well-def$^{\underline{d}}$ (ϕ multi-valued)

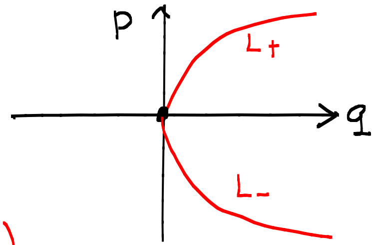
$\rightsquigarrow$ $I_\hbar(L, \iota, a)$ ✓

(3) Non-graph.



non-caustic points $\Rightarrow \sum_{\text{sum}}$ (layers)

· Eg.
$$\begin{array}{ccc} L & \overset{\iota}{\hookrightarrow} & T^*\mathbb{R} \\ \wr\wr & & \wr\wr \\ \mathbb{R} & & \mathbb{R}^2 \\ x & \longmapsto & (x^2, x) \end{array}$$



$$\alpha|_L = p\,dq = x\,dx^2 = d\underbrace{\left(\frac{2x^3}{3}\right)}_{\phi}$$

$\forall \quad a = B(x)\,|dx|^{1/2}$ half-density on L

$$\Longrightarrow (\pi_{L_\pm}^{-1})^* a = \frac{1}{\sqrt{2}}\, q^{-\frac{1}{4}}\, B(\pm q^{1/2})\, |dq|^{1/2} \qquad (x = q^{\frac{1}{2}})$$

$$\Longrightarrow \quad I_\hbar(L, \iota, a)(\underset{>0}{q})$$

$$\overset{\text{sum over}}{\underset{\text{layers}}{=\!=}} \left(e^{\frac{2}{3} i q^{3/2}/\hbar} B(q^{\frac{1}{2}}) + e^{-\frac{2}{3} i q^{3/2}/\hbar} B(-q^{\frac{1}{2}})\right) \frac{q^{\frac{-1}{4}}}{2} |dq|^{\frac{1}{2}}$$

Say $\quad H(q, p) = p^2 - q$

$$dH = 2p\,dp - dq = -\iota_{X_H}(dp \wedge dq)$$

$$\Longrightarrow \quad X_H = \frac{\partial}{\partial p} + 2p \frac{\partial}{\partial q} = \frac{\partial}{\partial x} \qquad \left(\begin{matrix} p = x \\ q = x^2 \end{matrix}\right)$$

$\xRightarrow{X_H\text{-inv.}} \quad B \equiv$ constant. (say 1)

$$\rightsquigarrow \quad I_\hbar = \left(e^{\frac{2}{3} i q^{3/2}/\hbar} + e^{-\frac{2}{3} i q^{3/2}/\hbar}\right) \frac{q^{\frac{-1}{4}}}{2} |dq|^{\frac{1}{2}} \quad \text{is}$$

semi-classical sol$^{\underline{n}}$ to $\quad -\hbar^2 \frac{\partial^2}{\partial x^2}\psi - x\psi = E\psi$

BUT $\quad I_\hbar(q = 0)$ blowup $\left(\because \lim_{q \to 0} q^{-\frac{1}{4}} = \infty\right)$.

T^*M, $\omega = d\alpha$ = -Curv. of S^1-bdl. Q_M

tiny circle

$$\frac{\mathbb{R}}{2\pi\hbar\mathbb{Z}} = S^1_\hbar \longrightarrow Q_M \xrightarrow{\ \pi\ } T^*M$$

trivial bundle w/ connection 1-form

$$\varphi = -\pi^*\alpha + d\sigma$$

$$(\sigma|_{S^1_\hbar} = d\theta)$$

$$S^1_\hbar \curvearrowright \mathbb{C}, \quad x \mapsto e^{-ix/\hbar}. \quad \overset{\text{assoc.}}{\underset{\text{bdl.}}{\rightsquigarrow}} \quad \mathbb{C} \longrightarrow \mathcal{E}_M \to T^*M$$

<u>Theorem</u> $L \overset{\iota}{\hookrightarrow} T^*M$ Lagr.

$$[\alpha|_L] \in H^1(L, 2\pi\hbar\mathbb{Z})$$

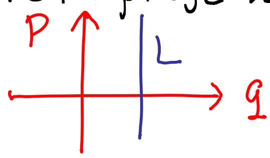
$\iff$ $\mathcal{E}_M|_L$ admits parallel section.

i.e. $(\mathcal{E}_M|_L, \nabla) \cong (L \times \mathbb{C}, d)$

§ Maslov correction.

$L \subset T^*\mathbb{R}$ NOT projectable to $\mathbb{R}_q$

Eg. 1. $L = \{q_0\} \times \mathbb{R}$



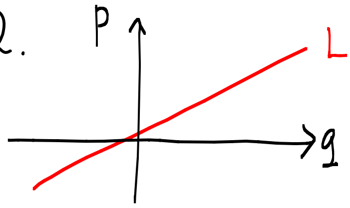
(i.e. position at q_0, but indeterminate momentum p.
$\rightsquigarrow$ probability = δ_{q_0} delta fu.)

'Try' project to p :

phase fu on L : $\tau(x) = -q_0 x$

$$(\pi_p^{-1})^* e^{i\tau/\hbar} |dx|^{1/2} = e^{-iq_0 p/\hbar} q_0 |dp|^{1/2}$$

$$= \mathcal{F}(\delta_{q_0})$$ Fourier transform !

Eg2. 

$L=\{p=kq\}$.   i.e.: $\iota:\mathbb{R}\longrightarrow T^*\mathbb{R}$

$\iota(x)=(x,kx)$

q-projection   p-projection

$$d(p\,dq)=\omega=d(-q\,dp)$$

⇒ phase fu. $\phi(x)=kx^2/2$ $\tau(x)=-kx^2/2$

half density on L, $a=A|dx|^{1/2}$, say $A\equiv$ const.

$(\pi_L^{-1})^*a=A|dq|^{1/2}$ $(\pi_p^{-1})^*a=|k|^{-1/2}A|dp|^{1/2}$

Quantizatⁿ. $I_\hbar=e^{ikq^2/2\hbar}A|dq|^{1/2}$ $\overset{\text{Fourier}}{\longleftarrow}$ $I_\hbar$

$$J_\hbar=\left(|k|^{\frac12}e^{-i\pi\,\mathrm{sgn}(k)/4}e^{ikq^2/2\hbar}\right)\left(|k|^{-\frac12}A|dq|^{1/2}\right)$$

$$=e^{-i\pi\,\mathrm{Sgn}(k)/4}\cdot I_\hbar$$

const. phase shift $e^{-i\pi\,\mathrm{Sgn}(k)/4}$

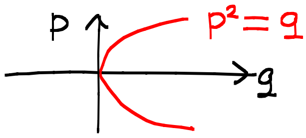
In general $L\subset T^*\mathbb{R}$, compare q- and p-projectⁿ

$$J_\hbar=e^{-i\pi\,\mathrm{sgn}(k)/4}I_\hbar+O(\hbar)$$

where $\phi=\tau+\iota^*(qp)$

$T:=\tau\circ\pi_p^{-1}\rightsquigarrow k(q):=T''(p(q))$

- Back to example $\begin{cases} L : & \iota : \mathbb{R} \longrightarrow T^*\mathbb{R} \\ & \iota(x) = (x^2, x) \\ a = B(x)\,|dx|^{1/2} & \text{on } L \end{cases}$



Use p-projection (stationary phase $\rightsquigarrow$)

$$J_\hbar = \left[e^{-i\frac{\pi}{4}} e^{2i\, q^{3/2}/3\hbar} B(-q^{1/2}) + e^{i\pi/4} e^{-2i q^{3/2}/3\hbar} B(q^{1/2}) \right] \frac{|dq|^{1/2}}{\sqrt{2}\, q^{1/4}} + O(\hbar)$$

(Recall
$$I_\hbar(L,\iota,a)(q) = \left(e^{\frac{2}{3} i\, q^{3/2}/\hbar} B(q^{\frac12}) + e^{-\frac{2}{3} i\, q^{3/2}/\hbar} B(-q^{\frac12})\right) \frac{q^{-\frac14}}{2} |dq|^{\frac12} \Big)$$

- Order 0^{th}: $I_\hbar = J_\hbar$ (up to phase)

  $I_\hbar$ singular, $J_\hbar$ smooth !! $\Rightarrow$ better.

- extra phase factor $e^{\mp i\pi/4}$

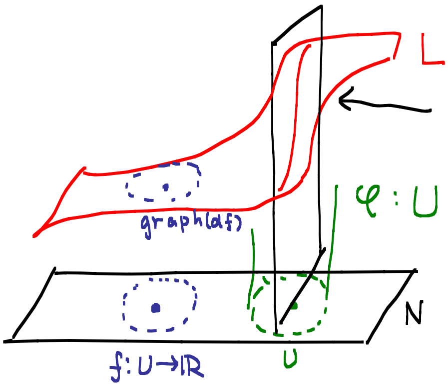
  $\Rightarrow$ need Maslov index $m(L)$ for $L \overset{\text{closed curve}}{\subset} T^*\mathbb{R}$

  s.t. $\frac{\pi\hbar}{2} m_L + \int_L p\,dq \in 2\pi\hbar\mathbb{Z}$

  (Maslov quantization condition).

Eg. Harmonic osc. $\Rightarrow$ $E \in (\mathbb{Z} + \frac{1}{2})\hbar$.

$L \xhookrightarrow{?} T^*M$ <u>non-projectable.</u>



$\Sigma \triangleq \{ d_{\mathbb{R}^m}\varphi = 0 \}$

$\cap$

$\varphi : U \times \mathbb{R}^m \longrightarrow \mathbb{R}$

fiber critical set.

$\overset{\text{loc.}}{\Longrightarrow}$ $L = \mathrm{Graph}(d\varphi|_{\Sigma})$

Construction of Lagrangians $L \subset T^*M$

Combine: exact graph + conormal bdl.

"Family version"

$$\begin{array}{ccccc} M & \xleftarrow{\pi} & B & \xrightarrow{\varphi} & \mathbb{R} \\ & & \downarrow & & \downarrow \\ C := \pi^* T^*M & \underset{\text{coisotropic}}{\subset} & T^*B & \underset{\text{Lagr.}}{\supset} & \underbrace{\mathrm{Graph}(d\varphi)}_{\triangleq L} \end{array}$$

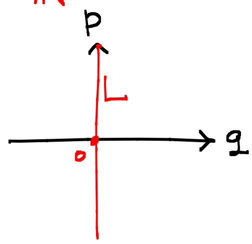
$$\Rightarrow \quad \begin{array}{ccc} \Sigma_\varphi & & T^*M \\ \sim \downarrow & & \parallel \\ L \cap C/\sim & \xhookrightarrow{\text{Lagr.}} & C/\sim \end{array}$$

$\rightsquigarrow$ Lagr. $\Sigma_\varphi \longrightarrow T^*M$)

Eg. For $L = \{ q = 0 \} = 0 \times \mathbb{R}^n \subset T^*\mathbb{R}^n$

Choose $\varphi : U \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$\varphi(q, \theta) = \sum_j q_j \theta_j \Rightarrow \Sigma = L$



Use Maslov ansatz

$$(2\pi\hbar)^{-n/2} \int_{\mathbb{R}^n} e^{i(\langle p, q\rangle + T(p))/\hbar} a(p)\, dp\, |dq|^{1/2}$$

For general $\varphi : U \times \mathbb{R}^m \longrightarrow \mathbb{R}$,

$$(2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} e^{i\phi(q,\theta)/\hbar} a(q,\theta)\, |d\theta|\, |dq|^{1/2}$$

- Inv. under coord. changes.

- In general, given

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & \mathbb{R} \\ {\scriptstyle \pi}\downarrow & & \\ M & & \end{array}$$

submersion π

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \uparrow \\ E & & E^* \\ \downarrow & & \uparrow \\ T_B & & T_B^* \\ \downarrow & & \uparrow \\ \pi^*T_M & & \pi^*T_M^* \\ \downarrow & & \uparrow \\ 0 & & 0 \end{array}$$

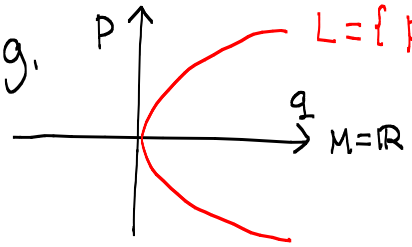
$d_{B/M}\varphi$ fiberwise derivative

$\Uparrow$

$\ni d\varphi$

$\Sigma \triangleq \{ d_{B/M}\varphi = 0 \} \subseteq B$ fiber critical set

Prop: φ non-degen. $\Rightarrow \Sigma \xhookrightarrow{d\varphi} T_M^*$ exact Lagr. emb.

E.g. 

$L=\{p^2=q\}$ from $\begin{array}{ccc} B=M\times\mathbb{R} & \xrightarrow{\varphi} & \mathbb{R} \\ \downarrow{\scriptstyle\pi} & & \\ M & & \end{array}$ $\qquad \varphi(x,\theta)=x\theta+\frac{\theta^3}{3}$

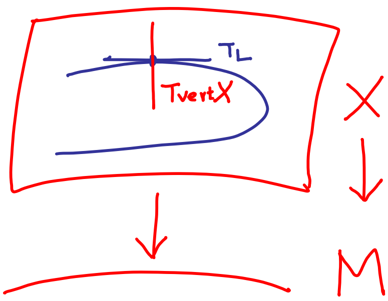
$$d_{B/M}\varphi = x+\theta^2$$

$\nabla(\ \text{—''—}\ ) = dx + 2\theta\, d\theta$ non-degen.

$\Sigma \quad = \{x=-\theta^2\} \xrightarrow[\sim]{d\varphi} L \subset T^*M$

- $\forall\ L \overset{\text{Lagr}}{\subset} T^*_M$, $\exists$ (loc.) $M \twoheadleftarrow B \xrightarrow{\varphi} \mathbb{R}$ realizing L,

unique up to 1) $\varphi + C$, 2) fiber diffeo of B, 3) $\times(\mathbb{R}, \theta^2)$.

Remark: Maslov object 

Lagr. $L \xhookrightarrow{i} X = T^*_M$

2 Lagr. subbdl. $\qquad$ sympl. v.b.

$\rightsquigarrow\ i_* T_L,\ T_{vert}X|_L \subset T_X|_L$

$\rightsquigarrow\quad \mu(L) \in H^1(L, \mathbb{Z}_4)$

$\rightsquigarrow$ flat line bundle $\mathbb{C} \to \mathcal{M} \to L$

w/ holonomy $\subseteq \mathbb{Z}_4$

§ WKB quantization

$$\begin{array}{ccc} & & \mathcal{E} \\ & & \mathbb{C}\downarrow \\ L & \xhookrightarrow{\iota} & T^*M \end{array} \quad F_{\mathcal{E}} = \omega_{can} \quad \rightsquigarrow \quad \begin{array}{c} \Phi \\ \mathbb{C}\downarrow \\ L \end{array} := \mathcal{M} \otimes \mathcal{E}|_L$$

phase bundle (flat).

- Say $\begin{array}{c} B \\ \pi\downarrow \\ M \end{array} \xrightarrow{\varphi} \mathbb{R} \rightsquigarrow \underbrace{\{ d_{B/M}\varphi = 0 \}}_{\Sigma} \xhookrightarrow[\sim]{d\varphi} L \subset T^*M$

$$S_\varphi := e^{i\varphi/\hbar} \in \Gamma(L, \Phi) \cap \operatorname{Ker} \nabla \quad \text{flat section}$$

$$0 \longrightarrow \underbrace{T_{vert}(B/M)}_{E} \longrightarrow TB \longrightarrow \pi^* T_M \longrightarrow 0$$

Def^{n}. amplitude

$$\alpha \in \Gamma(B, \underbrace{\underbrace{|\Lambda|^{\frac12} B \otimes |\Lambda|^{\frac12} E}_{|\Lambda|^{\frac12}\pi^* T_M \otimes |\Lambda| E}}_{\mathcal{A}\ \text{amplitude bdl.}})$$

$$\downarrow S_{B/M}$$

$$I_\hbar(\varphi, \alpha)(x) \in \Gamma(M, |\Lambda|^{\frac12} M)$$

$$= (2\pi\hbar)^{-\frac{n}{2}} e^{-in\frac{\pi}{4}} \left(\int_{\pi^{-1}(x)} e^{i\varphi/\hbar} \sigma_x \right) |dx|^{\frac12}$$

w/ $n = \dim B/M$

$\alpha = \pi^* |dx|^{\frac12} \otimes \sigma$

(from M to L need phase) $\quad S_\alpha \triangleq \alpha \otimes S_\varphi \overset{loc.}{\in} \Gamma(L, |\Lambda|^{\frac12} L) =: \mathcal{S}_L$

symbol space

Theorem: $L \xhookrightarrow{\imath} T^*M$ described (loc) by

i) $M \overset{\pi_j}{\twoheadleftarrow} B_j \xrightarrow{\varphi_j} \mathbb{R}$ and ii) amplitudes α_j on B_j, $j = 1, 2$

$$S_{\alpha_1} = S_{\alpha_2} \text{ on } L \iff I_\hbar(\varphi_1, \alpha_1) = I_\hbar(\varphi_2, \alpha_2) + O(\hbar)$$

[Pf: Stationary phase to fibers of $B_j \to M$.

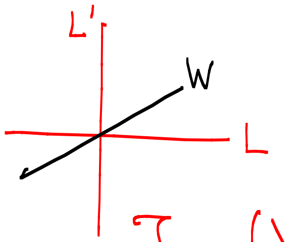
[caustic pt. $\rightsquigarrow$ $I_\hbar$ distribution

§ Maslov bundle.

(V, ω) Sympl. v.s.

$$L, L' \in \mathcal{L}(V) = \{\text{linear Lagr.}\} \simeq U(n)/O(n)$$

$$\mathcal{L}_{L,L'} = \{\text{linear Lagr. } W \text{ s.t. } W \pitchfork L \text{ \& } W \pitchfork L'\} \quad (\subseteq \mathcal{L}(V))$$



Consider

$$\mathcal{F}_{L,L'}(V) \triangleq \{ f : \mathcal{L}_{L,L'} \to \mathbb{Z} : f(W) - f(W') = \mathrm{ind}(L,L',W) - \mathrm{ind}(L,L',W') \ \ \forall\, W, W' \}$$

$\mathbb{Z}$ ↻ simply transitively

Hence, $\forall\ \underbrace{\lambda, \lambda'}_{\text{Lagr. subbdl.}} \leq \underbrace{E}_{\text{sympl. v.b}/M}$

$\overset{\text{def}^{n}.}{\rightsquigarrow}$ principal $\mathbb{Z} \longrightarrow M_{\lambda,\lambda'}(E) \longrightarrow M$

$$\mathcal{F}_{\lambda_x, \lambda'_x}(E_x) \longmapsto x$$

• $L \overset{\text{Lagr.}}{\hookrightarrow} T^*M = P \rightsquigarrow \underbrace{\lambda = T_L, \ \lambda' = T_v P|_L}_{\text{Lagr. subbdl.}} \leq \underbrace{TP|_L}_{\text{sympl. v.b.}}$

$\rightsquigarrow \mathbb{Z} \longrightarrow M_{\lambda,\lambda'}(E) \longrightarrow L$

Theorem: $M_{\lambda,\lambda'}(E) \simeq M_{L,i}$ (def^{d}. before).

Theorem. $\underbrace{\eta}_{\text{Coisotropic subbdl}}, \ \underbrace{\lambda, \lambda'}_{\text{Lagr. subbdl.}} \leq \underbrace{E}_{\text{sympl. V.B.}/M}$

Assume $\eta \cap \lambda$, $\eta \cap \lambda'$ constant ranks.

$$\Rightarrow \quad M_{\lambda,\lambda'}(E) \simeq \underbrace{M_{\lambda_\eta, \lambda'_\eta}(\eta/\eta')}_{\text{sympl. reduction.}}$$

§ Symplectic "category" $\mathcal{S}$

object: (P, ω) sympl. mfd.

morphism: $\mathrm{Hom}(P,Q) \triangleq \{ L \xhookrightarrow{\text{Lagr.}} \bar{P} \times Q \}$

$$\begin{tikzcd} \{ L \xhookrightarrow{\text{Lagr.}} \bar{P} \times Q \} \\ \{ P \xrightarrow[\text{sympl.}]{} Q \} \arrow[u, "\text{graph}"] \end{tikzcd}$$

"Composition": From Lagr. correspondence w/

kernel $P \times Q \times R \overset{\Delta}{\hookrightarrow} \bar{P} Q \bar{Q} R \bar{R} P$

(need clean intersections)

- $\mathcal{S} \overset{\text{full subcat.}}{\supset} \mathcal{S}_{cot} = \{ P = T^*M \}$

$$\mathcal{S}_{cot} \cup ((\mathrm{mfd})) = \{ M \}$$

same obj. but less Hom

- $C^{n+k} \overset{\text{coiso.}}{\subset} (P^{2n}, \omega)$

$\rightsquigarrow R_C^{n+k} \overset{\text{Lagr}}{\subset} \bar{P}^{2n} \times C/\!\sim^{2k}$ consists of $([c], c)$ w/ $c \in C$

i.e. $R_C \in \mathrm{Hom}(P, C/\!\sim)$

R_C epimorphism $R_C \circ R_C^* = 1_{C/\sim} \in \mathrm{End}(C/\!\sim)$

Define $K_C \triangleq R_C^* \circ R_C \in \mathrm{End}(P)$

$K_C^2 = K_C = K_C^*$ ($\sim$ ortho. proj.)

e.g. $L \overset{\text{Lagr.}}{\subset} P$, regard $L \in \mathrm{Hom}(pt., P)$

(assume $L \cap C$ cleanly).

$\rightsquigarrow L_C = R_C \circ L \in \mathrm{Hom}(pt, C/\!\sim)$ i.e. Lagr. in $C/\!\sim$

$L^C = K_C \circ L \in \mathrm{Hom}(pt., P)$ i.e. Lagr. in P

§ Sympl. mfd. & mechanics.

Classical $(P,\omega) \rightsquigarrow$ Lie alg. $(C^\infty(P), \{-,-\})$

w/ $\{f,g\} = X_g \cdot f$

s.t. $\{fh, g\} = f\{h,g\} + \{f,g\}h$

(i.e. Poisson alg. $C^\infty(P), \cdot, \{\ \}$)

- Hamilton eqt for observable $f \in C^\infty(P)$

$$\dot f = \{H, f\}$$

- f, g involutive $\overset{\Delta}{\Leftrightarrow}$ $\{f,g\} = 0$.

Quantum $\mathcal{H}_P$ Hilbert space (of states)

quantum observable : $A : \mathcal{H}_P \circlearrowleft$ self-adj. $A \in \underline{u}(\mathcal{H}_P)$

$(\underline{u}(\mathcal{H}_P), [\ \ ])$ Lie alg. w/ $[A, B] \triangleq \frac{i}{\hbar}(AB - BA)$

Schrödinger eqt. $\dot\psi = \frac{i}{\hbar} \hat H \psi$ w/ $\psi(t) \in \mathcal{H}_P$

Quantization: Dirac axioms:

$$\rho : C^\infty(P) \longrightarrow u(\mathcal{H}_P)$$

(i) $\rho(1) = 1_{\mathcal{H}_P}$, (ii) $\{\ \ \} \longrightarrow [\ \ ]$,

(iii) f_i's complete involutive $\Longrightarrow$ $\rho(f_i)$'s complete commuting

(i.e. $\forall g$ w/ $\{g, f_i\} = 0$ $\forall i$ $\Rightarrow g(x) = h(f_1(x), \dots, f_n(x))$)

(i.e. $\forall B \in u(\mathcal{H}_P)$ w/ $[B, \rho(f_i)] = 0$ $\forall i$ $\Rightarrow B = 1_{\mathcal{H}_P}$)

- May not exist.
- Weyl-von Neumann : Replace $\underset{\underset{\text{Lie Hamil}(P).}{\shortparallel}}{C^\infty(P)}$ by $\text{Sympl}(P)$

$\underline{u}(\mathcal{H}_P)$ by $U(\mathcal{H}_P)$

§ Geometric Quantization.

Prequantization: $\rho : C^\infty(P,\omega) \longrightarrow \mathfrak{u}(\mathcal{H}_P)$

s.t. Dirac axioms (i) & (ii)

(Segal) $P = T^*M$ & $\omega = d\alpha$

Pick $\mathcal{H}_P := L^2(P,\mathbb{C})$ wrt $\int_P f\bar{g}\,\omega^n$

$\rho(f) := -i\hbar X_f + (f - \alpha(X_f))$. ✓

(Kostant-Souriau) $\mathbb{C} \longrightarrow E \longrightarrow (P,\omega)$

conn. ∇ w/ curv. ω

$\mathcal{H} = \Gamma_{(2)}(E)$ & $\rho(f) = -i\hbar\nabla_{X_f} + f$.

Exact seq. of Lie alg:

$$0 \to \mathbb{R} \to \mathrm{Vect}(Q,\varphi) \to \mathrm{Vect}(P,\omega) \to H^1(P,\mathbb{R}) \to 0$$

$\bar{X}_f - fX$ (horiz. lift) $\uparrow$ f $\qquad \uparrow\cong$ $C^\infty(P)$

(Q: circle bdl of E & φ: conn. 1-form)

Central ext$^{\underline{n}}$ of groups:

$$1 \to S^1 \to \mathrm{Aut}(Q,\varphi) \to \mathrm{Aut}(P,\omega)$$

$\mathrm{Aut}(Q,\varphi) \xrightarrow{\text{by composition}} L^2(Q)$ preserving $\int_Q u\bar{v}\,\varphi\wedge(d\varphi)^n$

Eg. $\mathbb{R}^{2n}$ linear fcl. $\qquad q_i \qquad\qquad p_i$

$\rightsquigarrow$ Hamil. v.f. $\quad X_{q_i} = -\frac{\partial}{\partial p_i} \qquad X_{p_i} = \frac{\partial}{\partial q_i}$

(lift to $Q = \mathbb{R}^{2n}\times S^1$) $\quad \xi_{q_i} = -\frac{\partial}{\partial p_i} - q_i X \qquad \xi_{p_i} = \frac{\partial}{\partial q_i} \qquad \left(X = \frac{\partial}{\partial\theta}\right)$

(i.e. $\xi_f = \bar{X}_f - (f - \alpha(X_f))X$)

$$[\xi_{q_i}, \xi_{p_j}] = \xi_{\{q_i, p_j\}} = \delta_{ij} X$$

$$\Rightarrow \quad \mathfrak{h}_n := \mathbb{R}\langle \xi_{q_i}\text{'s}, \xi_{p_i}\text{'s}, X \rangle \underset{\text{Lie subalg.}}{\leq} \mathfrak{aut}(Q, \varphi)$$

$$\simeq \mathbb{R}^{2n} \times \mathbb{R}$$

Heisenberg $\quad [(v, a), (u, b)] = (0, \omega(u, v))$

$$\mathcal{H}_n = e^{\mathfrak{h}_n} \leq \mathrm{Aut}(Q, \varphi)$$

$$1 \to S^1 \to \mathcal{H}_n \longrightarrow \mathbb{R}^{2n} \longrightarrow 0$$

$$\hookrightarrow$$

$$\Gamma(P, E^{\otimes k}) \simeq \left\{ f \in C^\infty(Q) : f(p \cdot a) = e^{-\frac{ika}{\hbar}} f(p), \; \forall p \in Q, \forall a \in S^1 \right\}$$

$$\simeq \left(-\frac{ik}{\hbar}\right)\text{- eigenspace of } \; X : C^\infty(Q) \circlearrowleft$$

$$\nabla_\eta \longleftrightarrow \mathcal{L}_{\bar{\eta}} \qquad \bar{\eta} = \text{horizontal lift of v.f. } \eta \text{ on } P.$$

Fix k

$$C^\infty(P) \curvearrowright C^\infty(Q) \supseteq \Gamma(P, E^{\otimes k})$$

$$f \mapsto \xi_f^{(k)} := -\frac{i\hbar}{k} \xi_f, \quad \rho_k(f) = -\frac{i\hbar}{k} \overline{X}_f + f$$

- $\rho_k(\{f, g\}) \overset{\checkmark}{=} k\,[\rho_k(f), \rho_k(g)]$ i.e. Dirac (i)
- $\rho_k(f)$ self-adjoint ($\because$ Hamil. v.f. is integrable). ✓ i.e. Dirac (ii).

Polarization

Issue (Dirac (iii), completeness).

eg. $T^*\mathbb{R}$ $\quad \rho(q) = -i\hbar\frac{\partial}{\partial p} + q, \quad \rho(p) = -i\hbar\frac{\partial}{\partial q}$

($\hat{q}$ should just be $q\times$)

Require p - independence. $\quad C_q^\infty(\mathbb{R}^2) \simeq C^\infty(\mathbb{R})$

$$\rightsquigarrow \quad \rho(q) = q, \quad \rho(p) = -i\hbar\frac{\partial}{\partial q} \qquad \checkmark$$

WKB $\rightsquigarrow$

quantum states $\in |\Omega|_0^{1/2} M$

$\wr$ via metric / vol.

$C^\infty(M) = \Gamma_q(T^*M, E)$

i.e. sect^{2} of E, $\parallel$ along fibers of $T^*M \to M$

Def: Polarization = involution Lagr. subbdle. $\mathcal{F} \leq TP \otimes_{\mathbb{R}} \mathbb{C}$

Quantum state space $\quad \mathcal{H} := \{ s \in \Gamma(P, E) \mid \nabla_X s = 0 \ \forall\ X \leq \mathcal{F} \}$

Real polarization $\quad \mathcal{F} = \overline{\mathcal{F}}$

e.g. $\mathbb{R}^{2n}$ w/ q = const (or p = const)

$$\rightsquigarrow \quad f = \int_{\mathbb{R}^n_q} f(q)\,\delta(q)\,dq = \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^n_p} e^{i\pi\langle p,q\rangle/\hbar}\, \hat{f}(p)\, dp$$

$\underset{\text{pol.}}{q = \text{Const}} \Rightarrow \mathcal{H} = C^\infty(\mathbb{R}^n_q)$

$\underset{\text{pol.}}{p = \text{const}} \Rightarrow \mathcal{H} = \{ \psi \in \Gamma(\mathbb{R}^{2n}, E) \mid (\frac{\partial}{\partial q} - 2\pi i p)\psi = 0 \}$

i.e. $\psi(q,p) = v(p)\, e^{i\pi\langle p,q\rangle/\hbar}$

Theorem: (P, ω) w/ real pol. $\mathcal{F}$

- loc. given by $q \equiv$ const.

  i.e. $\exists$ (loc). $f_1, \dots, f_n \in C^\infty(P)$ s.t. $\mathcal{F} = \langle X_{f_1}, \dots, X_{f_n} \rangle$
- each leaf is affine.

Eg. Assume: $\pi_1(\text{leaf}) = 0$ & $P/\mathcal{F}$ smooth mfd.

say T^*M w/ $\omega_{can} + \pi^*\eta$ (twisted cotangent bdl.)

$D_{\mathbb{C}} := \mathcal{F} \cap \overline{\mathcal{F}}$, $E_{\mathbb{C}} := \mathcal{F} + \overline{\mathcal{F}}$ $\leq TP \otimes \mathbb{C}$

$= D \otimes \mathbb{C}$ $\qquad = E \otimes \mathbb{C}$ $\qquad D^{\perp} = E$

$\mathcal{F}$ inv. $\Rightarrow$ D inv. $\not\Rightarrow$ E inv.

"IF" E inv. ; P/D , P/E sm. mfd.

$P/D \xrightarrow{\pi} P/E$ submersion

$\Rightarrow$ fibers of π : Kähler

$\mathcal{H} \ni s$ sect^{s}/P s.t. · $\parallel$ along D
· holo. along π.

Complex polarization $D = 0$

i.e. $\mathcal{F} = \{(v, iJv) : v \in TP\}$ $\exists$ (integr.) cx. str. J on P

i.e. (P, J, ω) Kähler.

eg. $P = \mathbb{C}$ $\qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right)$

Identify $\qquad E \simeq \mathbb{C}$

$$\nabla_{\frac{\partial}{\partial \bar{z}}} = 2\frac{\partial}{\partial \bar{z}} + \underset{\sim}{z} \cdot$$

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \psi = 0 \rightsquigarrow \frac{\partial}{\partial \bar{z}} \log \psi = -\frac{z}{2}$$

$$\log \psi = -\frac{|z|^2}{2} + \text{holo.}$$

i.e. $\psi(z) = \varphi(z) e^{-|z|^2/2}$ s.t. $\bar{\partial}\varphi = 0$

$$\psi \in \Gamma_{(2)}(E) \Leftrightarrow \int_{\mathbb{C}} |\varphi|^2 \underbrace{e^{-|z|^2} dz d\bar{z}}_{\text{new measure}} < \infty$$

Remark: (P, ω) w/ cpx. pol. $\equiv$ $\exists$ Kähler str.

Assume P cpt., Riemann-Roch $\Rightarrow$ o

$$\dim \mathcal{H}_{P_k} = \dim H^0(P, E^{\otimes k}) \underset{k \gg 0}{=\!=} \chi(P, E^{\otimes k}) \underset{RR}{=\!=} \int_P e^{k\omega}\, \mathrm{Td}_P$$

$$= \int_P \left(1 + k\omega + k^2\frac{\omega^2}{2} + \dots + k^n \frac{\omega^n}{n!}\right)\left(1 + c_1(P) + \dots + \mathrm{Td}_n(P)\right)$$

$$= k^n \underbrace{\int_P \frac{\omega^n}{n!}}_{\mathrm{Vol}(P)} + O(k^{n-1}).$$

Metaplectic & Metalinean structure.

$\pi_1(Sp(2n,\mathbb{R})) \simeq \mathbb{Z}$ $\quad (\because Sp(2n,\mathbb{R}) \overset{h.e.}{\sim} U(n))$

$$\rightsquigarrow \quad \begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \underset{\text{connected.}}{Mp(n)} \arrow[r] & Sp(2n,\mathbb{R}) \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[u, equal] & M\ell(n) \arrow[r] \arrow[u, hook] & GL(n,\mathbb{R}) \arrow[r] \arrow[u, hook, "\text{(preserve a Lagr.)}"'] & 1 \end{tikzcd}$$

$M\ell(n) \simeq GL^+(n,\mathbb{R}) \times \mathbb{Z}_4$, $\quad (A,a) \mapsto Ae^{i\pi a}$

(P,ω) metaplectic str. $\overset{\Delta}{\Leftrightarrow}$ T_P lift from Sp to Mp

$\Leftrightarrow$ $w_2(P) = 0$

$\mathbb{R}^n \to L \to P$ metalinean str. $\overset{\Delta}{\Leftrightarrow}$ L lift from GL to $M\ell$

$\Leftrightarrow$ $w_1(L)^2 = 0$

$\underset{\text{via } \mathbb{Z}_4 \leqslant U(1)}{\rightsquigarrow}$ $\mathbb{C} \to \Lambda^{\frac{1}{2}}L \to P$ bdl. of half forms.

s.t. $(\Lambda^{\frac{1}{2}}L)^{\otimes 2} \simeq \det L \otimes \mathbb{C}$

- (P,ω) sympl. $\quad L \leqslant T_P$ Lagr. subbdl.

$\Rightarrow$ $T_P = L \oplus JL$ $\quad$ (wrt any compat. J)

$w_2(P) = w_1(L)^2$

So P metaplectic $\Leftrightarrow$ L metalinean.

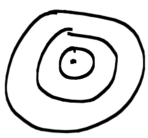
Def. If $\mathcal{F} (= L) \leqslant T_P$ inv. (i.e. real pol.),

Quantum state space $\mathcal{H}_{\mathcal{F}} \triangleq \Gamma(P, E \otimes \Lambda^{-\frac{1}{2}}\mathcal{F}) \cap \{\parallel \text{ along leaves}\}$

(distributional sense)

- $\exists$ natural inner product.

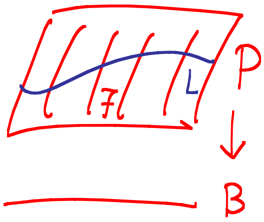
Eg. 1D SHO $\quad \mathbb{R}^2 \setminus 0$ w/ circle pol.



$\rightsquigarrow$ Bohr-Sommerfeld : $\frac{1}{2}\pi r_0^2 = \pi\hbar(n + \frac{1}{2})$

Quantization of semi-classical states.

Given $\mathbb{C} \to E \longrightarrow (P, \omega)$ metaplectic
w/ (good) real polarization $\mathcal{F}$



$\forall$ semi-classical state (L, s).

- If $L \pitchfork \mathcal{F}$ at ≤ 1 point,
  $s \in \Gamma(L, E \otimes \Lambda^{-\frac{1}{2}}\mathcal{F}) \rightsquigarrow$ extend by covariant const. along leaves
  $\rightsquigarrow \tilde{s} \in \mathcal{H}$
- $L \pitchfork \mathcal{F}$ multi-section $\rightsquigarrow$ superposition $\tilde{s} \in \mathcal{H}$
- L = a leaf
  $s \in \Gamma(L, E \otimes \Lambda^{-\frac{1}{2}}\mathcal{F})$
  $\nabla s = 0$
  $\Rightarrow$ it's a (distributional) elt. in $\mathcal{H}$ (say via pairing).
  $\sim \delta$-function.

Blattner-Kostant-Sternberg kernel.

$\mathcal{F}_1, \mathcal{F}_2$ 2 (good) real polarizations

Relate $\mathcal{H}_{\mathcal{F}_1}$ & $\mathcal{H}_{\mathcal{F}_2}$: (assume $\mathcal{F}_1 \pitchfork \mathcal{F}_2$)

$\omega \rightsquigarrow \mathcal{F}_1 \simeq \mathcal{F}_2^*$

half-forms $\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}$ of $\begin{matrix}\mathcal{F}_1\\ \mathcal{F}_2\end{matrix}$ $\overset{\text{pair}}{\rightsquigarrow}$ function $(\lambda_1, \lambda_2) : P \longrightarrow \mathbb{C}$

$\sigma_i \in \mathcal{H}_{\mathcal{F}_i} \sim \mathcal{F}_i$-$\parallel$ section $s_i \otimes \lambda_i$ of $E \otimes \Lambda^{\frac{1}{2}}\mathcal{F}_i$

$\rightsquigarrow \quad \mathcal{H}_{\mathcal{F}_1} \otimes \mathcal{H}_{\mathcal{F}_2} \xrightarrow{\langle\langle\ \rangle\rangle} \mathbb{C}$ non-degen.

$$\langle\langle \sigma_1, \sigma_2 \rangle\rangle = \frac{1}{(2\pi\hbar)^{n/2}} \int_P \langle s_1, s_2 \rangle_E \cdot (\lambda_1, \lambda_2)\, \omega^n$$

$$\rightsquigarrow \quad \mathcal{H}_{\mathcal{F}_1} \underset{\cong}{\xrightarrow{\langle\langle\ \rangle\rangle}} \mathcal{H}_{\mathcal{F}_2}^* \xrightarrow{\langle\ \rangle_{\mathcal{H}_2}} \mathcal{H}_{\mathcal{F}_2}$$